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GROUP ACTIONS ON FINITE ACYCLIC SIMPLICIAL COMPLEXES

BY

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ABSTRACT

In this paper we develop some homological techniques to obtain fixed points for groups acting on finite $\mathbb{Z}$-acyclic complexes. In particular we show that if a group G acts on a finite 2-dimensional acyclic simplicial complex D, then the fixed point set of G on D is either empty or acyclic. We supply some machinery for determining which of the two cases occurs. The Feit–Thompson Odd Order Theorem is used in obtaining this result.

0. Introduction

This paper is concerned with the action of a finite group G on a (abstract) finite simplicial complex D. In [8], Oliver showed that the assumption that D is $\mathbb{Z}$-acyclic does not restrict the homology $H_*(D^G;\mathbb{Z})$, where D^G is the fixed point subcomplex of D, except when G has a very specialized structure (cf. [8]). One motivation to our paper is the question of whether assuming in addition that D has low dimension does restrict $H_*(D^G;\mathbb{Z})$, in particular, what happens when D is two-dimensional? Recall that if D is a tree (i.e. one-dimensional acyclic complex) then D^G is a (nonempty) tree.

Another motivation to our paper is the question of whether one can define a broad enough class $\mathcal{D}$ of finite acyclic simplicial complexes having the property

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that $D^{<g>} \in \mathcal{D}$, for all $g \in G$, whenever $D \in \mathcal{D}$ and G is a group acting on D. Note that Quillen's conjecture in [7] asserts that if D is the order complex of the poset of nontrivial p-subgroups of a finite G, denoted by $\mathcal{S}_p(G)$, p a prime, then D^G is contractible, provided D is contractible. One can prove (see [2]) that Quillen's conjecture follows from the implication $\mathcal{S}_p(G)$ is acyclic $\rightarrow \mathcal{S}_p(G) \in \mathcal{D}$.

By taking the first barycentric subdivision of D, there is no loss of generality for our purposes to assume that G acts on D is such a way that if an element $g \in G$ fixes a simplex of D, it fixes all its vertices. In that event we call (D, G) an **admissible pair** (see the precise definitions in section 1 below). Our chain complexes have *coefficients in* $\mathbb{Z}$. We prove:

THEOREM 1: *Assume (D, G) is an admissible pair with D finite. If D is two–dimensional and acyclic, then*

(1) *D^G is either empty or acyclic.*
(2) *If G is solvable, D^G is acyclic.*

At the end of section 3 we give an example (see example 1) which was communicated to us by R. Oliver, of a finite 2-dimensional acyclic complex such that A_5 acts on it with no fixed points. We believe our example is the 2-skeleton of the so called 'Spherical dodecahedron space' (see, e.g. [9], p. 225) regarded as a CW-complex. Hence our Theorem 1 is at its best general form. We note that the Spherical Dodecahedron Space is a Poincare space (see [9], p. 225 for a definition). It is possible that other examples of fixed point free actions of finite groups on finite two–dimensional acyclic complexes are associated to Poincare spaces. However we believe that for most groups G acting on a finite two–dimensional acyclic complex D, D^G is acyclic. Indeed Theorem (3.2) and Lemma (3.5) are useful in showing this. Lemma (3.6) illustrates how to use Theorem (3.2) and Lemma (3.5) to show that if $G = A_n$, $n \geq 6$ acts on a 2-dimensional finite acyclic complex it fixes an acyclic subcomplex.

The proof of Theorem 1 requires various results on group actions on simplicial complexes proved in section 2. Theorem (2.3) generalizes a well known result on acyclic covers; e.g. [4], p. 92 or [3] Lemma (4.4) and the references therein. Theorem (2.7) deals with the top homologies of D^G, when (D, G) is an admissible pair and Theorem (2.4) gives a connection between $H_*(D^G)$, $H_*(D)$ and the order complex of the poset of all proper nontrivial subgroups of G, when (D, G) is an acyclic pair (definition in section 1).

We mention that there is still work to be done in determining precisely which groups G can act fixed point freely on finite two–dimensional acyclic complexes. Notice that Lemma (2.1) together with Theorem (3.4) reduce the question to the case when G is simple. Furthermore, we know of no example of a fixed point free action on finite 2-dimensional *contractible* complexes—our example is *not* contractible.

We mention that the proof of Theorem 1 does not require the Classification Theorem of finite simple groups but only the Feit–Thompson Odd Order Theorem.

1. Notation and preliminaries

We begin by establishing our basic notation and definitions. *Throughout D is a finite complex and G is a finite group.* Given a group G, $H < G$ means that H is a proper subgroup of G, while $H \leq G$ means that H is a subgroup of G (not necessarily proper). The same notation holds for sets, that is $A \subset B$ means that A is proper in B and $A \subseteq B$ means that A is a subset of B (not necessarily proper). Our simplicial complexes are abstract simplicial complexes as (for example) in [6], p. 15. All complexes in this paper are simplicial complexes. All homology groups in this paper are homology groups with *coefficients in* $\mathbb{Z}$. A simplicial map $\varphi : D \to L$ from a complex D to a complex L is a map of vertices such that $\{\varphi(v_0), ..., \varphi(v_k)\}$ is a simplex of L, for every k-simplex $\{v_0, ..., v_k\}$ of D. For a vertex v of D, write φv for its image under φ and for a simplex $\sigma = \{v_0, ..., v_k\}$ of D, write $\varphi\sigma$ for the simplex $\{\varphi v_0, ..., \varphi v_k\}$. We denote by $\varphi_{\#} : \mathcal{C}(D) \to \mathcal{C}(L)$ the chain map induced on the simplicial chain complexes. *We denote by $\partial : \mathcal{C}(D) \to \mathcal{C}(D)$ the boundary map.* φ is a simplicial isomorphism if φ is bijective on the vertices and $\varphi^{-1} : L \to D$ is a simplicial map. Write Aut(D) for the group of all simplicial automorphisms of D.

Given a poset P, the **order complex of** P is the simplicial complex whose simplices are finite chains. This complex will also be denoted here by P. We denote by P^*, the dual poset. Given a complex D we always view the first barycentric subdivision of D, denoted by sd(D), as a poset with the simplices of D as vertices and inclusion the order relation. Given two posets P and Q recall that the **join**, $P \vee Q$, of P and Q is the poset whose vertex set is the disjoint union of P and Q and whose order relation on P (resp. Q) is the same as in P (resp. Q) and any member of Q is larger than any member of P.

Given a group G write $\mathcal{S}(G)$ for the poset (and order complex) of all proper nontrivial subgroups of G.

Definition: Let $\mathcal{F}$ be a collection of nonempty sets, the **nerve** of $\mathcal{F}$, $N(\mathcal{F})$, is the simplicial complex whose vertex set is $\mathcal{F}$ and whose simplices are those subsets $\sigma \subseteq \mathcal{F}$ such that $\emptyset \neq F_\sigma = \bigcap_{F\in\sigma} F$. Let A be a set such that $A \subset F$, for all $F \in \mathcal{F}$. Define $N(\mathcal{F}, A)$ to be the subcomplex of $N(\mathcal{F})$ whose vertex set is $\mathcal{F}$ and whose simplices are those simplices $\sigma \in N(\mathcal{F})$ such that $\bigcap_{F\in\sigma} F$ properly contains A. Note that $N(\mathcal{F}) = N(\mathcal{F}, \emptyset)$. ∎

We record that

(1.1): *Let $\mathcal{F}$ be a collection of sets. Assume $\psi : \mathcal{F} \to \mathcal{F}$ is a map such that for each $F \in \mathcal{F}, F \subseteq \psi(F)$. Let A be a set such that $A \subset F$, for all $F \in \mathcal{F}$. Then*

(1) *$\psi : N(\mathcal{F}, A) \to N(\mathcal{F}, A)$ is a simplicial map.*

(2) *$\psi_* : H_*(N(\mathcal{F}, A)) \to H_*(N(\mathcal{F}, A))$ is the identity homomorphism.*

Proof: (1) is obvious. For (2) let $i : \mathcal{F} \to \mathcal{F}$ be the identity map. Then for each simplex $\{F_0, ..., F_k\}$ of $N(\mathcal{F}, A)$, $\{F_0, ..., F_k, \psi(F_0), ..., \psi(F_k)\}$ is a simplex of $N(\mathcal{F}, \mathcal{A})$, so by definition, ψ and i are contiguous. By [6], p. 67, $\psi_* = i_*$ so (2) follows. ∎

For completeness we recall the following definitions and result due to A. Björner [4]. Let P be a poset. A subset R of P is **initial** if for every $p \in P$ there exists $r \in R$, with $r \leq p$. R is **join coherent** if whenever a subset T of R has an upper bound in P, it has a join in P. Given an initial subset R of P define the complex $\Phi(P, R)$ on the vertex set R, by taking as simplices those finite nonempty subsets of R which have an upper bound.

(1.2) (A. Björner [4], p. 93): *Let R be a join coherent initial subset of a poset P. Then the order complex of P and $\Phi(P, R)$ have the same homotopy type.*

We thus define $\mathcal{S}_1(G) = \Phi(\mathcal{S}(G), \mathcal{S}(G))$ and we observe that

(1.3): *$\mathcal{S}(G)$ and $\mathcal{S}_1(G)$ have the same homotopy type.*

Proof: This is immediate from (1.2), since evidently $R = \mathcal{S}(G)$ is an initial set and if $\{H_0, ..., H_k\}$ have an upper bound, then $\langle H_0, ..., H_k\rangle$ is its join. ∎

We record the following two results

(1.4) (D. Quillen [7], p. 102): *Let $f, g : P \to Q$ be maps of posets such that $f(x) \leq g(x)$, for all $x \in P$, then $|f|$ and $|g|$ are homotopic.*

(1.5): *Let $\phi : P \to Q$ be a map of posets and assume that for every $p \in Q$, $\phi^{-1}((Q \geq q)) = \{p \in P : \phi(p) \geq q\}$ is acyclic. Then P and Q have the same homology.*

Proof: Since the proof is essentially the one given in [3], we give an outline of the proof. Define three acyclic carriers, as follows. $\Phi_{Q,P}$ which assigns to each simplex of Q an acyclic subcomplex of P, by $\Phi_{P,Q}(s) = \phi^{-1}((Q \geq \min(s)))$, where $\min(s)$ is the minimal element of s. Φ_P which assigns to each simplex t of P an acyclic subcomplex of P by $\Phi_P(t) = \Phi_{Q,P}(\phi(t))$ and Φ_Q which assigns to each simplex s of Q the subcomplex $(Q \geq \min(s))$. Set $f = \phi_\# : \mathcal{C}(P) \to \mathcal{C}(Q)$ and let $g : \mathcal{C}(Q) \to \mathcal{C}(\mathcal{P})$ be a chain map carried by $\Phi_{Q,P}$. Let i and j be the identity maps of P and Q respectively. The reader can verify that both $i_\#$ and $g \circ f$ are carried by Φ_P and both $j_\#$ and $f \circ g$ are carried by Φ_Q. So by the Acyclic Carrier Theorem ([6], p. 74), $\phi_* : H_*(P) \to H_*(Q)$ is an isomorphism. ∎

We will also need the following fact

(1.6): *Assume D, L are finite–dimensional complexes such that $\tilde{H}_i(D) = \tilde{H}_j(L) = 0$ for all $i \leq n$ and $j \leq m$, $n, m \geq -1$. Then $\tilde{H}_k(D \vee L) = 0$, for all $k \leq n+m+2$ and $\tilde{H}_{n+m+3}(D \vee L) = \tilde{H}_{n+1}(D) \otimes \tilde{H}_{m+1}(L)$.*

Proof: See e.g. [1]. ∎

Finally recall that a **cover** of a complex D is a collection of subcomplexes $\mathcal{F}$ of D such that each simplex of D is a simplex of some member $F \in \mathcal{F}$.

2. Some aspects of admissible action

Definition: A pair (D, G) is an **admissible pair** if D is a complex, $G \leq \mathrm{Aut}(D)$ and for every simplex $\sigma = \{v_0, \dots, v_k\}$ of D, if $g\sigma = \sigma$, then $gv_i = v_i$, for all $i = 0, \dots, k$. ∎

Given an admissible pair (D, G) and $H \in \mathcal{S}(G)$ write D^H for the full subcomplex of D with vertex set the fixed point set of H on D. Denote $\mathrm{Fix}(D, G) = \mathrm{Fix}(D)$ the subcomplex of D, whose vertex set is the set $\bigcup_{H \in \mathcal{S}(G)} D^H$ and whose simplices are subsets σ such that σ is a simplex of D and $\sigma \subseteq D^H$, for some $H \in \mathcal{S}(G)$.

(2.1): *Let $H \lhd G$. Assume $\mathcal{D}$ is a family of finite acyclic complexes such that for all $D \in \mathcal{D}$, if (D,X) is an admissible pair, then $D^X \in \mathcal{D}$, for $X = H$ and for $X = G/H$. Then, for all $D \in \mathcal{D}$, if (D,G) is an admissible pair, then $D^G \in \mathcal{D}$.*

Proof: This follows immediately from the fact that $D^G = (D^H)^{G/H}$. ∎

Definition: An admissible pair (D,G) is **proper** if $D^G \subset D^H$, for all $H \in \mathcal{S}(G)$. (D,G) is **acyclic** if D^H is acyclic, for all $H \in \mathcal{S}(G)$. ∎

(2.2): *Let (D,G) be an admissible pair. Let $\mathcal{M} \subseteq \mathcal{S}_1(G)$ be a nonempty subset regarded as a* **full** *subcomplex and assume that*

(1) *For every simplex s of $\mathcal{M}$, $D^G \subset D^{<s>}$.*

Let $\mathcal{F} = \{D^H : H \in \mathcal{M}\}$ and $D_0 = D^G$. Then $\mathcal{M}$ and $N = N(\mathcal{F}, D_0)$ have the same homotopy type.

Proof: Define $\phi : \mathrm{sd}(\mathcal{M}) \to \mathrm{sd}(N)$ and $\psi : \mathrm{sd}(N) \to \mathrm{sd}(\mathcal{M})$ as follows. For a simplex s of $\mathcal{M}$, $\phi(s) = \{D^H : H \in s\}$ and for a simplex σ of N, $\psi(\sigma) = \{H \in \mathcal{M} : \bigcap_{F\in\sigma} F \subseteq D^H\}$. By hypothesis (1), σ and ψ are well defined. Now if $t \subseteq s$ are simplices of $\mathrm{sd}(\mathcal{M})$, then $\phi(t) \subseteq \phi(s)$ and if $\tau \subseteq \sigma$ are simplices of $\mathrm{sd}(N)$, then $\psi(\tau) \subseteq \psi(\sigma)$. Further $s \subseteq \psi \circ \phi(s)$ and $\sigma \subseteq \phi \circ \psi(\sigma)$. Hence the lemma follows from (1.4). ∎

Definition: Let D_0 be a subcomplex of a complex D. A **mod D_0 acyclic cover** of D is a cover $\mathcal{F}$ of D such that $D_0 \subset F$, for all $F \in \mathcal{F}$, each member of $\mathcal{F}$ is acyclic and for any nonempty subset σ of $\mathcal{F}$, $F_\sigma = \bigcap_{F\in\sigma} F$ is acyclic, or $F_\sigma = D_0$. ∎

(2.3) THEOREM: *Let D_0 be a subcomplex of the complex D. Let $\mathcal{F}$ be a mode D_0 acyclic cover of D. Set $N = N(\mathcal{F}, D_0)$. Then D has the same homology as the join $N \vee D_0$.*

Proof: Let $P = \mathrm{sd}(D_0)$, $Q = \mathrm{sd}(N)$, $R = \mathrm{sd}(D)$ and $J = Q \vee P^*$. Define a map $f : R^* \to J$, as follows:

(1) If σ is a simplex of D_0, $f(\sigma) = \sigma$.

(2) Otherwise, $f(\sigma) = \{F \in N : \sigma \text{ is a simplex of } F\}$.

We first show that f is order preserving. Let σ, τ be two simplices of D with $\sigma \geq_{R^*} \tau$, that is $\sigma \subseteq \tau$. Assume τ is a simplex of D_0, then $f(\sigma) = \sigma \subseteq \tau = f(\tau)$.

So $f(\sigma) \geq_J f(\tau)$. Assume σ is not a simplex of D_0, then clearly $f(\tau) \subseteq f(\sigma)$ and again $f(\sigma) \geq_J f(\tau)$. Finally if σ is a simplex of D_0 and τ is not, evidently $f(\sigma) >_J f(\tau)$.

We claim that for all $j \in J$, $D_j = f^{-1}(J(\geq j))$ is acyclic so (1.5) completes the proof. Now if j is a simplex of D_0, then D_j is just the set of all faces of j. If j is a simplex of N, then a simplex σ of D is in D_j iff $f(\sigma) \geq_J j$, then either σ is a simplex of D_0, or σ is not a simplex of D_0 and σ is a simplex of F, for all $F \in j$. Thus D_j is the set of all simplices of D contained in $\bigcap_{F \in j} F$, so by hypothesis it is acyclic. ∎

As a corollary we have the following result

(2.4) THEOREM: *Let (D, G) be a proper acyclic pair. Assume $\mathcal{S}(G) \neq \emptyset$. Then* Fix$(D)$ *has the same homology as the join $\mathcal{S}(G) \vee D^G$.*

Proof: Choose $\mathcal{M} = \mathcal{S}_1(G)$ in (2.2). Then as (D, G) is proper, $\mathcal{M}$ satisfies hypothesis (2.2.1) so by (2.2), $\mathcal{S}_1(G)$ has the same homology as the complex $N(\mathcal{F}, D^G)$. Now Theorem (2.3) completes the proof. ∎

Let (D, G) be an admissible pair.

Notation: From now on we fix the letter n to denote the dimension of D. ∎

Given a k-chain $c \in C_k(D)$, the **support** of c is the support of c written in terms of the canonical basis, i.e., the k-oriented simplices, for some fixed orientation.

Assume $H_n(D) = H_{n-1}(D) = 0$. As $\partial : C_n(D) \to C_{n-1}(D)$ has trivial kernel, for every cycle $z \in Z_{n-1}(D)$, there exists a unique chain $c \in C_n(D)$ such that $z = \partial(c)$. We set $c = \partial^{-1}(z)$ and we define

Definition: For a cycle $z \in Z_{n-1}(D)$, the **bounding number** of z is $\sum_{i=1}^{m} |n_i|$, where $\partial^{-1}(z) = \sum_{i=1}^{m} n_i \sigma_i$. ∎

We also define

Definition: The **bounding support** of a cycle $z \in Z_{n-1}(D)$ is the support of the n-chain $\partial^{-1}(z)$. ∎

(2.5): *Let (D, G) be an admissible pair. Assume that $n \geq 2$ and*

(1) $H_n(D) = H_{n-1}(D) = 0$.

(2) *For any cycle $z \in Z_{n-1}(D^G)$ there exists an (oriented) n-simplex σ in the bounding support of z such that σ is a simplex of D^G. Then $H_n(D^G) = H_{n-1}(D^G) = 0$.*

Proof: Clearly $H_n(D^G) = 0$. Assume $H_{n-1}(D^G) \neq 0$. Let $z \in Z_{n-1}(D^G)$ be a cycle that doesn't bound in $\mathcal{C}(D^G)$. Choose z to have minimal bounding number. Let $c = \partial^{-1}(z)$. By hypothesis there exists an n-simplex σ of D^G such that σ is in the support of c. We may assume without loss that the coefficient of σ in c is positive. Evidently $z - \partial(\sigma)$ is an $(n-1)$-cycle in $\mathcal{C}(D^G)$ and $\partial(c-\sigma) = z - \partial(\sigma)$. Hence the bounding number of $z - \partial(\sigma)$ is smaller then that of z. By the choice of z, $c - \sigma$ is in $C_n(D^G)$, hence c is, a contradiction.

We will see in a minute that if (D, G) is an admissible pair and $H_n(D) = H_{n-1}(D) = 0$, then hypothesis (2) of (2.5) is satisfied. Before that we need

(2.6): *Let (D, G) be an admissible pair with G cyclic of prime order p. Assume $n \geq 2$ and $H_n(D) = H_{n-1}(D) = 0$. Let $z \in Z_{n-1}(D^G)$ be a cycle such that there exists no n-simplex σ in the bounding support of z such that σ is a simplex of D^G. Then $z = pz_1$, for some cycle $z_1 \in Z_{n-1}(D)$.*

Proof: Set $G = \langle g \rangle$. Let $[v_0, ..., v_{n-1}]$ be in the support of z. Let $c = \partial^{-1}(z)$. Then the simplices in the support of c having $[v_0, ..., v_{n-1}]$ as a face sum up in c in the following way: $n_1([v_0, \ldots, v_{n-1}, w_1] + [v_0, \ldots, v_{n-1}, gw_1] + \cdots + [v_0, \ldots, v_{n-1}, g^{p-1}w_1] + n_2([v_0, \ldots, v_{n-1}, w_2] + \cdots + [v_0, \ldots, v_{n-1}, g^{p-1}w_2]) + \cdots + n_k([v_0, \ldots, v_{n-1}, w_k] + \cdots + [v_0, \ldots, v_{n-1}, g^{p-1}w_k])$. Hence the coefficient in z of $[v_0, \ldots, v_{n-1}]$ is $(-1)^n p \sum_{i=1}^k n_i$ as asserted. ∎

(2.7) THEOREM: *Assume (D, G) is an admissible pair and $\tilde{H}_n(D) = \tilde{H}_{n-1}(D) = 0$. Then $\tilde{H}_n(D^G) = \tilde{H}_{n-1}(D^G) = 0$.*

Proof: First note that we may assume $n \geq 2$, since if $n = 0, 1$, D is a tree in which case, as was mentioned above, D^G is a tree.

Assume the theorem holds when G is cyclic of prime order. Then using induction, we see that the theorem holds for any cyclic group G. Let $z \in Z_{n-1}(D^G)$. Then $\partial^{-1}(z) \in C_n(D^{<g>})$, for all $g \in G$. Hence $\partial^{-1}(z) \in C_n(D^G)$ as asserted. So $H_{n-1}(D^G) = 0$. Obviously $H_n(D^G) = 0$.

Hence we may assume that G is cyclic of prime order. We show that (D, G) satisfies the hypotheses of (2.5). Assume false and let $z \in Z_{n-1}(D^G)$ such that no simplex in the bounding support of z lies in D^G. Pick such z with minimal bounding number. By (2.6), $z = pz_1$, for some $z_1 \in Z_{n-1}(D^G)$. Notice that the bounding number of z is p times the bounding number of z_1. By the choice of z, there exists a simplex in the bounding support of z_1 which is a simplex of D^G. But the bounding support of z equals the bounding support of z_1, a contradiction.

3. Two dimensional acyclic complexes

In this section D is two-dimensional acyclic complex. Our ongoing hypothesis is that D is finite and G is finite.

(3.1) THEOREM: *Assume (D, G) is an admissible pair. Then*

(1) $H_1(D^G) = H_2(D^G) = 0$.

(2) *If G is solvable, D^G is acyclic.*

Proof: (1) follows immediately from Theorem (2.7). For (2), using an obvious induction we may assume G is cyclic of prime order p. As is well known (see e.g. [5], chapter III), D^G is $\mathbb{Z}_p$ acyclic and in particular, $\tilde{H}_0(D^G) = 0$. Hence (2) follows from (1). ∎

HYPOTHESIS A: *$\mathcal{Z} \subseteq \mathcal{S}_1(G)$ is a subset such that:*

(1) $<\mathcal{Z}>= G$.

(2) *For any proper nonempty $\mathcal{Z}_1 \subset \mathcal{Z}$, $<\mathcal{Z}_1> \quad < G$.*

(3.2) THEOREM: *Assume that (D, G) is an admissible pair and that there exists a subset $\mathcal{Z} \subseteq \mathcal{S}_1(G)$ such that*

(1) *$\mathcal{Z}$ satisfies Hypothesis A.*

(2) *For any proper nonempty $\mathcal{Z}_1 \subset \mathcal{Z}$, $D^{<\mathcal{Z}_1>}$ is acyclic.*

Then if $|\mathcal{Z}| > 2$, D^G is either empty or acyclic and if $|\mathcal{Z}| > 3$, D^G is acyclic.

Proof: Let $\mathcal{Z}$ denote also the full subcomplex of $\mathcal{S}_1(G)$, with vertex set $\mathcal{Z}$. Assume that D^G is not acyclic. Set $|\mathcal{Z}| = n + 2$. Then the dimension of $\mathcal{Z}$ is n and $\tilde{H}_n(\mathcal{Z}) \simeq \mathbb{Z}$, while $\tilde{H}_k(\mathcal{Z}) = 0$, for $k \neq n$. Let $\mathcal{F} = \{D^H : H \in \mathcal{Z}\}$ and $L = \bigcup_{H \in \mathcal{Z}} D^H$. Notice that as $D^G \subset D^{<\sigma>}$ for each simplex σ of $\mathcal{Z}$, (2.2) says that $N(\mathcal{F}, D^G) = N$ has the same homotopy type as $\mathcal{Z}$. Since $\mathcal{F}$ is a cover of L, Theorem (2.3) says that L has the same homology as the join $N \vee D^G$ that is

(*) L has the same homology as $\mathcal{Z} \vee D^G$.

If $D^G \neq \emptyset$, then by (3.1), $\tilde{H}_0(D^G) \neq 0$ and by (1.6),

(**) $H_{n+1}(L) \neq 0$.

Notice now that L is of dimension ≥ 2 and L is a subcomplex of an acyclic complex of dimension two. Hence

(***) $H_k(L) = 0$, for $k > 1$.

As $n \geq 1$, (**) and (***) contradict each other.

Assume $D^G = \emptyset$ and $n \geq 2$. By (*), L has the same homology as $\mathcal{Z}$ so as $H_n(\mathcal{Z}) \neq 0$, (***) supplies a contradiction. ∎

The next lemma shows that every finite (nonabelian) simple group G satisfies Hypothesis A, for some $\mathcal{Z}$, with $|\mathcal{Z}| > 2$. We use the Feit–Thompson Odd Order Theorem.

(3.3): *Assume G is a finite nonabelian simple group. Then there exists a subset $\mathcal{Z} \subseteq \mathcal{S}_1(G)$ such that $|\mathcal{Z}| > 2$ and $\mathcal{Z}$ satisfies Hypothesis A.*

Proof: Let Inv(G) be the set of involutions in G. As G is simple $G =< \text{Inv}G >$. Hence there exists a subset $\mathcal{Z} \subseteq \text{Inv}(G)$ such that $G =< \mathcal{Z} >$ and $\mathcal{Z}$ satisfies Hypothesis A2. Since G is simple, $|\mathcal{Z}| > 2$, and we are done. ∎

(3.4) THEOREM: *Assume (D, G) is an admissible pair such that G is finite. Then D^G is either empty or acyclic.*

Proof: Let G be a minimal counter example. By minimality of G, D^H is acyclic, for all $H \in \mathcal{S}_1(G)$. Hence, by minimality of G, G is simple. By (3.3), G satisfies the hypotheses of (3.2) and hence by (3.2), D^G is either empty or acyclic, a contradiction. ∎

The following example shows that case $D^G = \emptyset$ of Theorem (3.4) can occur.

Example 1: Let D be the two-dimensional simplicial complex on the vertex set $V = \{a, b, c, d, e, f\} \cup \{1, 2, 3, 4, 5\} \cup \{(12), (13), (14), (15), (23), (24), (25), (34), (35), (45)\}$ so that $|V| = 21$. The (oriented) 2-simplices of D are as in Table I. ∎

Table I

a	b	c	d	e	f
[a, 1, (15)]	[b, 1, (15)]	[c, 1, (14)]	[d, 1, (12)]	[e, 1, (15)]	[f, 1, (12)]
[a, 1, (12)]	[b, 1, (13)]	[c, 1, (13)]	[d, 1, (13)]	[e, 1, (14)]	[f, 1, (14)]
[a, 2, (12)]	[b, 2, (24)]	[c, 2, (23)]	[d, 2, (24)]	[e, 2, (24)]	[f, 2, (12)]
[a, 2, (23)]	[b, 2, (25)]	[c, 2, (25)]	[d, 2, (12)]	[e, 2, (23)]	[f, 2, (25)]
[a, 3, (23)]	[b, 3, (13)]	[c, 3, (13)]	[d, 3, (13)]	[e, 3, (23)]	[f, 3, (34)]
[a, 3, (34)]	[b, 3, (34)]	[c, 3, (23)]	[d, 3, (35)]	[e, 3, (35)]	[f, 3, (35)]
[a, 4, (34)]	[b, 4, (34)]	[c, 4, (45)]	[d, 4, (45)]	[e, 4, (14)]	[f, 4, (14)]
[a, 4, (45)]	[b, 4, (24)]	[c, 4, (14)]	[d, 4, (24)]	[e, 4, (24)]	[f, 4, (34)]
[a, 5, (45)]	[b, 5, (25)]	[c, 5, (25)]	[d, 5, (35)]	[e, 5, (35)]	[f, 5, (25)]
[a, 5, (15)]	[b, 5, (15)]	[c, 5, (45)]	[d, 5, (45)]	[e, 5, (15)]	[f, 5, (35)]

Further, every k-simplex is a face of a 2-simplex, for $k = 0, 1$. This complex is obtained as follows: Let $G = A_5$ act on $\{1,2,3,4,5\}$. Let $X = G(5)$, the stabilizer in G of 5, $Y = G(\{4,5\})$, the global stabilizer in G of $\{4,5\}$ and $Z = N_G(<(12345)>)$, the normalizer in G of $<(12345)>$. so $X \simeq A_4$, $Y \simeq S_3$ and $Z \simeq D_{10}$. We take as vertices all right cosets of H in G, where $H \in \{X, Y, Z\}$ and as (oriented) 2-simplices $\{[Zg, Xg, Yg] : g \in G\}$. Every k-simplices is a face of a 2-simplex, for $k = 0, 1$. Clearly G acts simplicially on this complex via right multiplication.

We show this complex is acyclic. First note it has 21 vertices, 80 1-simplices and 60 2-simplices, so its Euler characteristic $\chi(D) = 1$. Next it is easily seen this complex is connected. Finally we sketch a proof that $H_1(D) = 0$. Denote by $\sigma_{t,i}$, where $t \in \{a, b, ..., f\}$ and $i \in \{1, ..., 10\}$, the simplex in column t and row i of Table I.

For $t \in \{a, b, c, d, e, f)$ set

$$c_t = \sum_{i=1}^{10} (-1)^{i+1} \sigma_{t,i}.$$

First we note that to show that every 1-cycle bounds it suffices to show that every cycle of the form:

$$(*) \qquad [v_0, v_1] + [v_1, v_2] + \cdots + [v_{k-1}, v_0]$$

bounds, where $v_0v_1\cdots v_{k-1}v_0$ is a closed path in the graph of D. Next observe that the distance from t to any vertex of D, in the graph of D is ≥ 2, for all $t \in \{a, ..., f\}$. Hence it suffices to show that 1-cycles of the form

$$(**) \qquad [t, v_1] + [v_1, v_2] + [v_2, v_3] + [v_3, v_4] + [v_4, t]$$

bound, where $tv_1v_2v_3v_4t$ is a closed path in the graph of D. Next it is not difficult to see that to show that cycles of the form (**) bound, it suffices to show that cycles of the form

$$(***) \qquad [i, (ij)] + [(ij), j] + [j, t] + [t, i]$$

bound, where $t \in \{a, b, ..., f\}$, $i < j$ are in $\{1, ..., 5\}$ and t is not adjacent to (ij) in the graph of D. Next note that G acts transitively on cycles of the form (***) so it suffices to show one of them bounds. We choose

$$x = [1, (12)] + [(12), 2] + [2, b] + [b, 1].$$

Set

$$y = [b, 1, (15)] - [b, 2, (25)] + [b, 5, (25)] - [b, 5, (15)] - c_c + c_d - c_e.$$

We leave it for the reader to verify that $\partial(y) = x$ and the example is complete.

(3.5): *Assume (D, G) is an admissible pair. Let $\mathcal{M} \subseteq S_1(G)$ be a nonempty subset and consider the subcomplex of $S_1(G)$ (also denoted by $\mathcal{M}$) with vertex set $\mathcal{M}$ and simplices $s \subseteq \mathcal{M}$ such that $D^{<s>} \neq \emptyset$. Assume $D^H \neq \emptyset$, for all $H \in \mathcal{M}$. Set $L = \bigcup_{H\in\mathcal{M}} D^H$ and $\mathcal{F} = \{D^H : H \in \mathcal{M}\}$. Then*

(1) *$\mathcal{M}$ and the nerve of $\mathcal{F}$, $N(\mathcal{F})$, have the same homotopy type.*
(2) *$\mathcal{M}$ and L have the same homology.*

Proof: The proof of (1) proceeds exactly as the proof of (2.2). For (2) note that $\mathcal{F}$ is a cover of L and that for each subset $\{F_{i_1}, F_{i_2}, ..., F_{i_k}\} \subseteq \mathcal{F}, \cap_{j=1}^{k} F_{i_j}$ is either empty or acyclic. Hence Theorem (2.3) completes the proof. ∎

(3.6): *Assume (D, G) is an admissible pair such that $G \simeq A_n$, $n \geq 6$. Then D^G is acyclic.*

Proof: Assume first that the lemma holds for $n = 6$. We proceed by induction on n. Let $n > 6$. We use Theorem (3.2) and we choose $\mathcal{Z} = \{H_1, H_2, H_3, H_4\}$, where $H_1 = \mathrm{Alt}\{1,...,n-3\}$, $H_2 =< (1,2,n-2) >$, $H_3 =< (1,2,n-1) >$ and $H_4 =< (1,2,n) >$. Then $H_{ijk} =< H_i, H_j, H_k > \simeq A_t$, for some $t < n$ and H_{ijk} is contained in some subgroup of G isomorphic to A_{n-1}, so by induction hypothesis (2) of Theorem (3.2) is satisfied and the lemma follows.

It remains to prove the lemma for $n = 6$. Assume $D^G = \emptyset$. We first show that

(*) If $H < G$, with $H \simeq A_5$, then $D^H = \emptyset$.

Assume $H = \mathrm{Alt}\{1,...,5\}$ and $D^H \neq \emptyset$. Then using Theorem (3.2), with $H_i =< (i,5,6) >$, $1 \leq i \leq 4$, we easily get a contradiction. Applying an outer automorphism we get (*).

Now we can use (3.5). Let $a = (12)(36)$ $b = (12)(45)$ $c = (12)(34)$ $d = (25)(36)$ $e = (26)(35)$. Then $[a,b] = [e,d] = 1$ and $ab = (36)(45)$ $de = (23)(56)$. Further, $|ac| = |ad| = |eb| = |ec| = |bc| = |cd| = |dab| = |bde| = 3$ so $< a,b,c > \simeq < a,b,d > \simeq < b,d,e > \simeq < c,d,e > \simeq S_4$ and $< a,c,d > \simeq < b,c,e > \simeq 3^2 : 2$. Moreover, $< a,e > \simeq D_{10}$ and $< b,c,d > \simeq A_6$. Let

$$\mathcal{M} = \{< a >, < b >, < c >, < d >, < e >\}$$

regarded as a subcomplex of $\mathcal{S}_1(G)$ as in (3.5). Then using (*) it is easily verified that $H_2(\mathcal{M}) \neq 0$, so by (3.5), $H_2(L) \neq 0$, where $L = \bigcup_{H \in \mathcal{M}} D^H$. But L is a subcomplex of D, a contradiction. ∎

(3.7): *Assume G is isomorphic to an alternating group A_n, $n \geq 6$, or to a group of Lie-type and Lie-rank ≥ 2, then G satisfies hypothesis A, for some $\mathcal{Z} \subseteq \mathcal{S}_1(G)$, with $|\mathcal{Z}| \geq 4$.*

Proof: If $G \simeq A_n$, $n \geq 6$, Pick $\mathcal{Z} = \{< (123) >, < (124) >, ..., < (12n) >\}$. So assume G is isomorphic to a group of Lie-type and Lie-rank ≥ 2. Let Σ be a root system for G and let $\Pi = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be the simple system. For $\alpha \in \Sigma$, let U_α denote the root group of α. Then $\mathcal{Z} = \{U_\alpha : \pm\alpha \in \Pi\}$ has the required properties as for each $\alpha \in \pm\Pi$, $\mathcal{Z} - \{U_\alpha\}$ is contained in a proper parabolic of G. ∎

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